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Random walks on invasion percolation clusters

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Abstract. Results are presented of numerical simulations of random walks on invasion percolation clusters in two and three dimensions, including simulations on continuum percolation clusters at threshold. Walks of up to 10^5 steps were taken. The results indicate that the fracton dimension of invasion percolation clusters, both on regular lattices and in the continuum, is the same, within numerical errors, as the fracton dimension of ordinary percolation clusters at threshold.

1. Introduction

Several numerical simulations of the diffusion of random walks on percolation clusters have been carried out [1-7], using de Gennes' idea of 'the ant in the labyrinth' [8]. In these simulations an 'ant', starting at an arbitrary site of a percolation cluster, randomly selects one of its nearest neighbours on the cluster and either moves there, if the site is occupied, or stays put, if the site is vacant. Whatever the decision, the time step is incremented by one unit. The process is repeated at each subsequent time step t and the ant traces out a random walk on the cluster.

There are two different approaches which may be taken: either to prepare a lattice realisation by randomly occupying the sites, with probability p, and then to allow the ant to start its walk on any cluster chosen randomly, or else to restrict the ant's walk to a single cluster, usually the incipient infinite cluster grown by the Leath method [9]. In this paper we consider the case when the ant is restricted to move on invasion percolation clusters, which resemble the incipient infinite cluster of percolation [10, 11].

It is of interest to study the asymptotic power law for the average squared displacement of the ant $\langle R^2 \rangle$ at time t:

$$\langle R^2 \rangle \sim t^{2/d_w}.$$

The diffusion is known to be anomalous at the percolation threshold [5, 6, 12, 13].

Another exponent of interest is the fracton dimension $\tilde{d}$ which is related to the exponent d_w via

$$\tilde{d} = 2D/d_w$$

where D is the fractal dimension of the incipient infinite cluster.

The fracton dimension was introduced by Alexander and Orbach [14] in a discussion of the density of the vibrational states on fractals. They conjectured that $\tilde{d} = \frac{4}{3}$ is an exact relation for percolation.

If the AO conjecture is true it implies that the dynamical exponents of percolation can be expressed in terms of the static exponents. For example, we would have

$$d_w = \tfrac{3}{2} D.$$

The exactness of the AO conjecture has been a matter of some controversy [15-18]. However, very high accuracy Monte Carlo simulations in two dimensions (see [1] and references within) and, more recently, in three dimensions [2], have shown small deviations from the AO values.

2. Numerical simulations

The present investigations were carried out on invasion percolation clusters in two and three dimensions. It was hoped that it would be possible to extend the calculations of Ben Avraham and Havlin [5, 6] for walks on single clusters because invasion percolation clusters can be grown to any size and hence longer walks can be taken without encoutering edge effects. To some extent these hopes were realised, although in the two-dimensional case the simulation ran into problems for walks greater than 10^4 steps for reasons which are not clearly understood by this author.

'Invasion percolation' was first introduced by Lenormand [19] and Chandler *et al* [20] in the context of a simulation of oil displacing water in a porous medium. It can be used as a method for growing single clusters on lattice structures. Wilkinson and Barsony [10] have shown that an interesting relationship exists between invasion percolation clusters and the incipient infinite cluster of ordinary percolation at threshold. By numerical simulations of invasion percolation they managed to obtain estimates of the critical percolation threshold and the gap exponent and fractal dimension of ordinary percolation in two and three dimensions. Within numerical errors, their results support the conjecture that invasion percolation reproduces ordinary percolation at threshold. However, this conjecture has not been shown to hold rigorously and is still somewhat mysterious [11]. The present investigations give it further support by providing estimates of dynamical scaling exponents for invasion percolation clusters.

Furthermore, these investigations include a simulation of random walks on the incipient infinite cluster of continuum percolation. The continuum percolation clusters were grown by the method described in [21]. Using this method one can grow single clusters at threshold without knowing what the exact value of p_c is and without having an underlying regular lattice structure. It also allows one to define nearest neighbours for the sites in the continuum. We were able to make more extensive simulations than those carried out by Wagner and Balberg [22] but, because of the problems which arose in all our 2D simulations for $t > 10^4$, the net results were no more accurate.

All the simulations involved averaging over several random walks on several different clusters. The clusters were grown so that their average radius was much larger (typically two or three times larger) than the average radius of the random walk at long times. The quantities which were measured were the following [2, 3, 5, 7].

(i) The square of the radius of the walk at time step t, R_t^2. This provided an estimate of the exponent d_w of the walks via the relation:

$$\langle R_t^2 \rangle \sim t^{2/d_w}.$$

(ii) The average number of returns to the origin at step t, $P_0(t)$. This provided a direct estimate of the fracton dimension of the walks via the relation:

$$P_0(t) \sim t^{-\tilde{d}/2}.$$

(iii) The number of distinct sites visited by the walk at step t, S_t. This gave an estimate of the fracton dimension via the relation:

$$\langle S_t \rangle \sim t^{\tilde{d}/2}.$$

3. Discussion of results

3.1. Two dimensions

In two dimensions, simulations of random walks were carried out on invasion percolation clusters on square and triangular lattices. The clusters were grown to a size of 100 000 sites, which was generous since there was an upper bound of 1500 sites over all the walks for the number of distinct sites visited by a walk. For the square lattice, 286 walks of 10^5 steps were executed on each of 54 clusters, making a total of 15 444 walks. For the triangular lattice, 35 clusters were used and the total number of walks executed was $286 \times 35 = 10\,010$.

A graph of $\log\langle R_t^2 \rangle^{1/2}$ against $\log t$ for the triangular lattice is shown in figure 1. There are deviations from the straight line behaviour at small t, as expected, but also at t bigger than 10^4, which was not expected. It is unlikely that the deviations at large t are due to the walks hitting the edge of the cluster since this was specifically excluded in the computer program and was the reason why the clusters were grown to such a large size. Possibly it is just that more statistics are needed for very long walks.

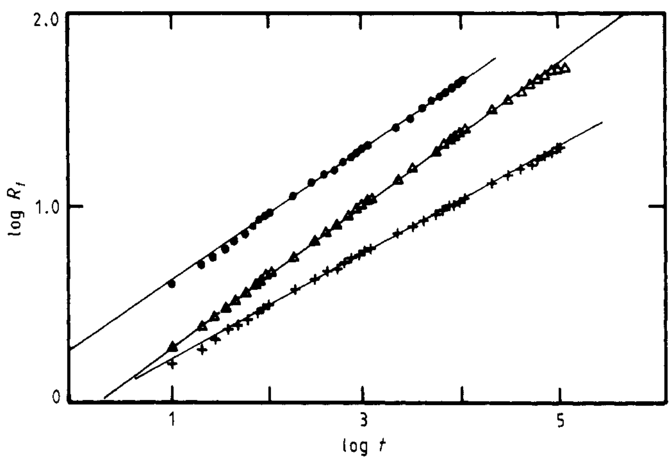

Figure 1. This shows graphs of $\log R_t$ against $\log t$ for random walks on invasion percolation clusters on the triangular lattice (△), cubic lattice (+) and in the continuum (●) where $R_t = \langle R_t^2 \rangle^{1/2}$.

The best value for the equilibrium slope of the graph is 2.82 ± 0.09. The best value for the square lattice was 2.87 ± 0.08. (The errors are calculated by dividing the straight line up into sections and seeing how the slope varies.) Ben Avraham and Havlin [6] reported a value of 2.84 ± 0.05 and the most accurate simulation known to the author [1] gives a value of 2.866 ± 0.009. The value corresponding to the AO conjecture is $\frac{3}{2}D \approx 2.844$.

A similar analysis was carried out for the graph of $\log\langle S_t\rangle$ against $\log t$. It was found that the best values for the equilibrium slopes of the graphs were 0.64 ± 0.05 for the triangular lattice and 0.62 ± 0.05 for the square lattice. Ben Avraham and Havlin reported a value of 0.63. These results are smaller than the asymptotic AO value of $\frac{2}{3}$. However, as discussed in [6], one expects them to be smaller because they depend on the value of the fractal dimension which is effectively smaller for a finite cluster.

Figure 2 shows a graph of $\log P_0(t)$ against $\log t$ for the square lattice. The results at small t were improved by carrying out an additional simulation of 100 000 walks of 10^2 steps. The best value for the slope of the graph is -0.64 ± 0.03. The same value was obtained for the triangular lattice.

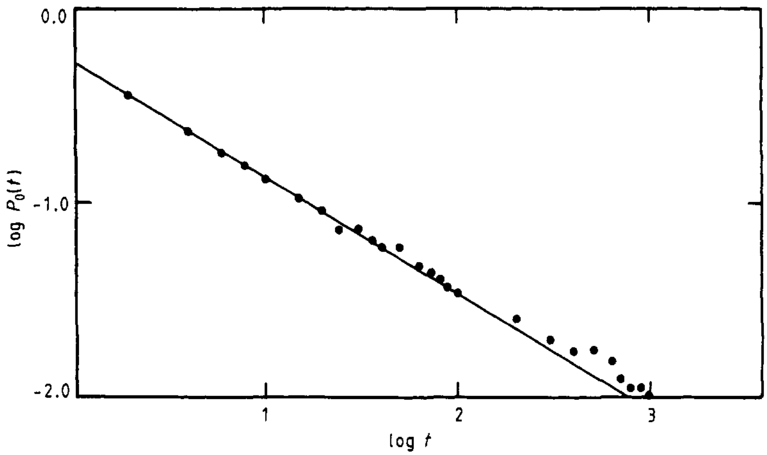

Figure 2. This shows a graph of $\log P_0(t)$ against $\log t$ for the square lattice.

3.2. *Three dimensions*

In three dimensions, random walks were carried out on invasion percolation clusters on a cubic lattice. The clusters were grown into a lattice of fixed extension and their size varied from about 20 000 to 50 000 sites. 300 walks of 10^5 steps were executed on each of 50 clusters, making a total of 15 000 walks.

The results for three dimensions were smoother than those for two dimensions. Figure 1 shows a graph of $\log\langle R_t^2\rangle^{1/2}$ against $\log t$. From these results, the asymptotic value of the slope was estimated to be 3.65 ± 0.09, which compares favourably with the value of 3.68 ± 0.05 [6] and the AO value of ≈ 3.73. Note that there is no obvious downturn for large t, as happened in the two-dimensional case.

The same sort of analysis can be done to estimate the asymptotic value of the slope of the graph of $\log\langle S_t\rangle$ against $\log t$. The value obtained is 0.63 ± 0.03.

Similarly, for the graph of $\log P_0(t)$ against $\log t$ the estimated slope is -0.64 ± 0.04.

3.3. *Continuum percolation clusters*

The simulation of random walks on continuum percolation clusters was slightly different from that for clusters on regular lattices. The continuum clusters were grown using the technique described in [21]. This technique involves the formulation of the continuum percolation of equal-sized disks in two dimensions as a bond percolation problem on a random lattice [23]. Because of the introduction of the random lattice it becomes possible to define 'nearest neighbours' for sites on continuum percolation clusters. When growing the clusters, information was stored about the coordinates of the sites, the number of their nearest neighbours and which of their nearest neighbours were on the cluster. When simulating a random walk on a continuum cluster the choice for the ant's next step was decided in the following way.

Assume that the ant is at site i at time t.

Choose at random a number r between 1 and the total number of nearest neighbours of site i.

If r is greater than the number of nearest neighbours of site i which are on the cluster, then the ant remains where it is at time $t+1$.

Otherwise, the ant moves to the rth site in the list of nearest neighbours of site i which are on the cluster.

This algorithm is somewhat different from the one used by Wagner and Balberg [22] in which, because they have no well defined nearest neighbours, they use M, the maximum number of intersections per disk in the particular sample studied, as a normalising constant. In their algorithm, if a given circle has N intersecting neighbours, the ant will move to each of these neighbours with probability $1/M$ and stay put with probability $(1-N)/M$. As they note, this leads to a slower diffusion on average than in usual lattice algorithms although their results show that it does not affect the asymptotic value for d_w.

The continuum clusters were grown to a size of 30 000 sites. A total of 16 600 walks of 10^4 steps were carried out, using 14 different clusters. (In the simulations of Wagner and Balberg, in which the ant was allowed to move on finite clusters as well as the infinite cluster, the total number of sites was 10 000, meaning that their clusters were of a considerably smaller radius than ours. We had hoped that this would enable us

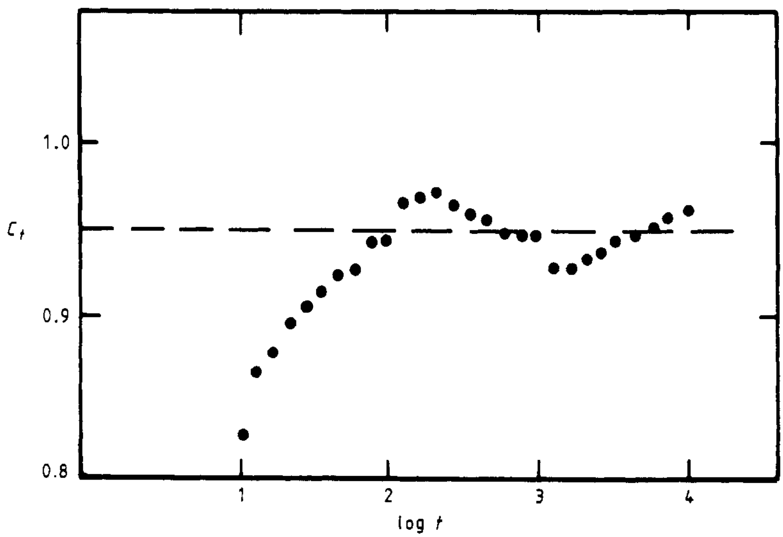

Figure 3. This shows a graph of C_t against t for random walks on continuum percolation clusters where $C_t = R_t/t^{1/d_w}$.

to consider longer walks; however, in the light of our experiences for the square and triangular lattice cases we did not feel that we could trust results for $t > 10^4$.)

The same quantities were calculated as in the regular lattice case.

Figure 1 shows a graph of $\log\langle R_t^2\rangle^{1/2}$ against $\log t$. The estimated value of the equilibrium slope of this graph is 2.90 ± 0.15. The value given by Wagner and Balberg is 2.87 ± 0.13.

Figure 3 shows a graph of $\langle R_t^2\rangle^{1/2}/t^{1/2.90}$ against t. It can be seen how the results oscillate about their equilibrium value, giving rise to the large error in the estimate.

An analysis of the results for $\langle S_t\rangle$ shows that the best estimate for the slope of the graph of $\log\langle S_t\rangle$ against $\log t$ is 0.62 ± 0.06.

The estimate obtained for the slope of the graph of $\log P_0(t)$ against $\log t$ is -0.62 ± 0.06.

These results should be compared with those obtained in the two-dimensional case on square and triangular lattices. The results from the simulation of random walks on continuum clusters are not as accurate as those on regular lattices. However, they seem to show the same average behaviour.

4. Conclusion

These simulations indicate that the fracton dimension of invasion percolation clusters, both on regular lattices and in the continuum, is the same, within the stated errors, as the fracton dimension of ordinary percolation clusters at threshold.

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